

# Quantum Dynamics in a Time-dependent Hard-Wall Spherical Trap

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Exact solution of the Schrödinger equation is given for a particle inside a hard sphere whose wall is moving with a constant velocity. Numerical computations are presented for both contracting and expanding spheres. The propagator is constructed and compared with the propagator of a particle in an infinite square well with one wall in uniform motion.

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## I. INTRODUCTION

Solving the Schrödinger equation with time-dependent boundary conditions, including moving ones, is a very hard work and can only be done in a few cases. See, *e.g.*, [1–3] and [4] for a recent review. Very interesting effects are seen in such problems; diffraction in time [5] is just such an instance. This phenomenon is characterized by quantum temporal oscillations in matter waves released from a confining region. It was shown when a wall acting as a perfect mirror, moves with finite velocity along the direction of propagation of a beam, the visibility of the fringes is enhanced [6]. Moshinsky's theoretical work has been extended to the case of particles which are suddenly released from a 1D box [7] and to particles with angular momentum [8] initially trapped in a hard spherical box. Exact solutions of the Schrödinger equation for a particle in a 1D box with a moving wall have been found [2, 3]. Using the semiclassical approximation, Luz and Cheng [10] evaluated the exact propagator of the problem. Grosche [11] did this task independently by means of an exact (mid-point) summation of the perturbation series with point-like perturbations.

The motivation of the present work is that sudden removal of the boundary is an idealized case, thus we consider a moving boundary instead of a sudden removal of the boundary. The limit of infinite velocity of the moving boundary clearly corresponds to the sudden removal case. Possible applications to optical effects connected with moving mirrors are additionally encouraging. For these reasons we aim to solve the Schrödinger equation for a particle in a hard sphere with varying radius.

## II. EXACT SOLUTION

Consider a particle with mass  $\mu$  inside a hard sphere with a time-dependent radius  $L(t)$ . The potential energy function is zero if  $r < L(t)$  and infinite otherwise. The

Schrödinger equation is then

$$i\hbar \frac{\partial}{\partial t} \Psi(\mathbf{r}, t) = -\frac{\hbar^2}{2\mu} \nabla^2 \Psi(\mathbf{r}, t), \quad (1)$$

with the boundary condition  $\Psi(\mathbf{r}, t)|_{r=L(t)} = 0$ .

The *instantaneous* energy eigenfunctions and eigenvalues are respectively

$$u_{lnm}(\mathbf{r}, t) = \sqrt{\frac{2}{L^3(t)}} \frac{1}{|j_{l+1}(x_{ln})|} j_l\left(x_{ln} \frac{r}{L(t)}\right) Y_{lm}(\theta, \phi) \quad (2)$$

$$E_{ln}(t) = \frac{\hbar^2 x_{ln}^2}{2\mu L^2(t)}, \quad (3)$$

$l = 0, 1, 2, \dots$ ;  $n = 1, 2, 3, \dots$  and  $m = -l, -l+1, \dots, l-1, l$ , where  $j_l(x)$  and  $Y_{lm}(\theta, \phi)$  are respectively spherical Bessel functions and harmonics.  $x_{ln}$  is the  $n^{\text{th}}$  zero of the spherical Bessel function of order  $l$ , *i.e.*,  $j_l(x_{ln}) = 0$ . It must be noted that all Bessel functions with  $l \neq 0$  have a zero at the origin, but to have a non-zero wave function these zeros must be excluded.

Using the method of "separation of variables" for solving the partial differential equation (1), we propose the solution

$$\Psi(\mathbf{r}, t) = \frac{U(r, t)}{r} Y_{lm}(\theta, \phi), \quad (4)$$

where we have used the spherical symmetry of the Hamiltonian.

Putting eq. (4) into eq. (1) one gets

$$i\hbar \frac{1}{r} \frac{\partial U(r, t)}{\partial t} = -\frac{\hbar^2}{2\mu} \left[ \frac{1}{r} \frac{\partial^2 U(r, t)}{\partial r^2} - \frac{l(l+1)}{r^2} \frac{U(r, t)}{r} \right]. \quad (5)$$

The radial part of the proposed wave-function,  $R(r, t) = U(r, t)/r$ , must be zero on the shell, thus the boundary conditions on  $U(r, t)$  are  $U(r, t)|_{r=0} = 0 = U(r, t)|_{r=L(t)}$ .

Now, we follow [2] to solve the eq. (5). By defining a new coordinate

$$s = \frac{r}{L(t)}, \quad (6)$$

we get

$$i\hbar \frac{\partial U(s, t)}{\partial t} = i\hbar \frac{\dot{L}(t)}{L(t)} s \frac{\partial U(s, t)}{\partial s} - \frac{\hbar^2}{2\mu} \frac{1}{L^2(t)} \times \left[ \frac{\partial^2 U(s, t)}{\partial s^2} - \frac{l(l+1)}{s^2} U(s, t) \right], \quad (7)$$

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where  $\dot{L}(t) = dL(t)/dt$  and moving boundary conditions are replaced by fixed-boundary ones;  $U(s, t)|_{s=0} = 0 = U(s, t)|_{s=1}$ . When the transformation

$$U(s, t) = \sqrt{\frac{2}{L(t)}} \exp \left[ \frac{i\mu}{2\hbar} L(t) \dot{L}(t) s^2 \right] \varphi(s, t) . \quad (8)$$

is introduced in eq. (7), one obtains

$$i\hbar \frac{\partial \varphi(s, t)}{\partial t} = -\frac{\hbar^2}{2\mu} \frac{1}{L^2(t)} \left[ \frac{\partial^2 \varphi(s, t)}{\partial s^2} - \frac{l(l+1)}{s^2} \varphi(s, t) \right] \quad (9)$$

for the uniform motion of the wall, *i.e.*,  $\ddot{L}(t) = 0$ . Boundary conditions on  $\varphi(s, t)$  are  $\varphi(s, t)|_{s=0} = 0 = \varphi(s, t)|_{s=1}$ . Defining the new time variable  $\tau$  as

$$\tau(t) = \int_0^t \frac{dt'}{L^2(t')}, \quad \Rightarrow \quad \frac{\partial}{\partial t} = \frac{1}{L^2(t)} \frac{\partial}{\partial \tau}, \quad (10)$$

Eq. (9) transforms to

$$i\hbar \frac{\partial \varphi(s, \tau)}{\partial \tau} = -\frac{\hbar^2}{2\mu} \left[ \frac{\partial^2 \varphi(s, \tau)}{\partial s^2} - \frac{l(l+1)}{s^2} \varphi(s, \tau) \right] \quad (11)$$

Inserting  $\varphi(s, \tau) = \exp(-iE'\tau/\hbar)\psi(s)$  in (11), one gets

$$E'\psi(s) = -\frac{\hbar^2}{2\mu} \left[ \frac{\partial^2 \psi(s)}{\partial s^2} - \frac{l(l+1)}{s^2} \psi(s) \right]. \quad (12)$$

By introducing new variable  $k^2 = 2\mu E'/\hbar^2$ , we obtain

$$\frac{\partial^2 \psi(s)}{\partial s^2} + \left( k^2 - \frac{l(l+1)}{s^2} \right) \psi(s) = 0. \quad (13)$$

The solutions of this equation are spherical Bessel functions

$$\psi(s) = s[c_1 j_l(ks) + c_2 n_l(ks)]. \quad (14)$$

If the radial wave-function  $R(r)$  is finite at the origin,  $c_2 = 0$ . The requirement that  $\psi(s) = 0$  at  $s = 1$  means that  $k$  can take on only those special values

$$k_{ln} = x_{ln} \quad (n = 1, 2, 3, \dots). \quad (15)$$

For the uniform change of the radius with velocity  $u$

$$L(t) = a + ut, \quad (16)$$

where  $a$  is the initial radius, one has

$$\tau(t) = \frac{t}{a(a+ut)}. \quad (17)$$

By using equations (17), (16), (15), (8) and (6) one obtains

$$R_{ln}(r, t) = c_1 \sqrt{\frac{2}{L(t)}} \exp \left[ \frac{i\mu}{2\hbar} u \frac{r^2}{L(t)} - i \frac{\hbar}{2\mu} x_{ln}^2 \frac{t}{aL(t)} \right] \times j_l \left( x_{ln} \frac{r}{L(t)} \right), \quad (18)$$

for the radial part of the wave-function. Unknown coefficient  $c_1$  is determined by the normalization condition

$$\int_0^{L(t)} dr r^2 \int d\Omega |\Psi_{lnm}(\mathbf{r}, t)|^2 = 1, \quad (19)$$

where

$$\Psi_{lnm}(\mathbf{r}, t) = R_{ln}(r, t) Y_{lm}(\theta, \phi) \quad (20)$$

are the solutions of the Schrödinger equation (1) for a particle in a spherical box with a wall in uniform motion and  $\int d\Omega = \int_{-1}^1 d(\cos \theta) \int_0^{2\pi} d\phi$ .

Using the orthogonality of the spherical Bessel functions [9]

$$\int_0^1 ds s^2 j_l(x_{ln}s) j_l(x_{lm}s) = \frac{1}{2} [j_{l+1}(x_{ln})]^2 \delta_{nm}, \quad (21)$$

one obtains

$$|c_1|^2 = \frac{1}{L^2(t)} \frac{1}{[j_{l+1}(x_{ln})]^2}. \quad (22)$$

Thus apart from a phase factor, one obtains

$$\begin{aligned} \Psi_{lnm}(\mathbf{r}, t) &= \frac{1}{L(t)} \sqrt{\frac{2}{L(t)}} \frac{1}{|j_{l+1}(x_{ln})|} \\ &\times \exp \left[ \frac{i\mu}{2\hbar} u \frac{r^2}{L(t)} - i \frac{\hbar}{2\mu} x_{ln}^2 \frac{t}{aL(t)} \right] \\ &\times j_l \left( x_{ln} \frac{r}{L(t)} \right) Y_{lm}(\theta, \phi) \\ &\equiv \exp \left[ i\alpha \xi(t) \left( \frac{r}{L(t)} \right)^2 - i x_{ln}^2 \frac{1 - 1/\xi(t)}{4\alpha} \right] \\ &\times u_{lnm}(\mathbf{r}, t), \end{aligned} \quad (23)$$

where we have introduced new dimensionless parameters  $\alpha = \mu a u / (2\hbar)$  and  $\xi(t) = L(t)/a$ .

Functions  $\Psi_{lnm}(\mathbf{r}, t)$  vanish at  $r = L(t)$ , remain normalized as the radius changes, and form a complete orthogonal set. The general solution of eq. (1) is a superposition of functions (23)

$$\Psi(\mathbf{r}, t) = \sum_{l'=0}^{\infty} \sum_{n'=1}^{\infty} \sum_{m'=-l'}^{l'} c_{l'n'm'} \Psi_{l'n'm'}(\mathbf{r}, t), \quad (24)$$

with time-independent coefficients  $c_{l'n'm'}$  determined from the relation

$$c_{l'n'm'} = \int_0^a dr r^2 \int d\Omega \Psi_{l'n'm'}^*(\mathbf{r}, 0) \Psi(\mathbf{r}, 0), \quad (25)$$

General solution can also be expanded in terms of instantaneous eigenfunctions as

$$\Psi(\mathbf{r}, t) = \sum_{l'=0}^{\infty} \sum_{n'=1}^{\infty} \sum_{m'=-l'}^{l'} b_{l'n'm'}(t) u_{l'n'm'}(\mathbf{r}, t), \quad (26)$$

now with time-dependent coefficients  $b_{l'n'm'}(t)$  determined from the relation

$$b_{l'n'm'}(t) = \int_0^{L(t)} dr r^2 \int d\Omega u_{l'n'm'}^*(\mathbf{r}, t) \Psi(\mathbf{r}, t), \quad (27)$$

Using eqs. (27) and (24) and the orthogonality of spherical harmonics, one finds

$$b_{l'n'm'}(t) = \frac{2}{|j_{l'+1}(x_{l'n'})|} \sum_{n''=1}^{\infty} c_{l'n''m'} \frac{1}{|j_{l'+1}(x_{l'n''})|} \times \exp \left[ -ix_{l'n''}^2 \frac{1 - 1/\xi(t)}{4\alpha} \right] I_{l'n''n'}^*(t, \alpha) \quad (28)$$

where

$$I_{l'n''n'}(t, \alpha) = \int_0^1 ds s^2 e^{-i\alpha\xi(t)s^2} j_{l'}(x_{l'n'}s) j_{l'}(x_{l'n''}s) \quad (29)$$

This integral is not elementary and following the procedure of [8], can be reduced to a combination of terms involving the Ferensel integrals and derivative of Legendre polynomials.

The expectation value of the energy of the particle is obtained from

$$\langle E(t) \rangle = \sum_{l'n'm'} |b_{l'n'm'}(t)|^2 E_{l'n'm'}(t). \quad (30)$$

If the particle is initially in an energy eigenstate, *i.e.*,  $\Psi(\mathbf{r}, 0) = u_{l1m}(\mathbf{r}, 0)$ , then

$$c_{l'n'm'} = \delta_{ll'} \delta_{mm'} \frac{2}{|j_{l+1}(x_{ln})| |j_{l+1}(x_{ln'})|} I_{lnn'}(0, \alpha), \quad (31)$$

which is not an unexpected result as quantum numbers  $l$  and  $m$  do not change.

### III. NUMERICAL CALCULATIONS

Numerical computations are shown in figs. 1 and 2 for a particle that is initially in the first excited state with three fold degeneracy. In this case we have

$$\frac{\langle E(t) \rangle}{E_{11m}(t)} = \sum_{n'} |b_{1n'm}(t)|^2 \left( \frac{x_{1n'}}{x_{11}} \right)^2. \quad (32)$$

for the ratio of energy expectation value to the instantaneous first excited state energy.

Figure 1 shows the squares of energy eigenfunction expansion coefficients versus  $\xi(t)$  for three different contraction rates  $\alpha$ . For these values of  $\alpha$ , it was found that series (28) converges for the first ten terms.

Figure 2 shows the ratio of the expectation value of the energy to the energy the particle would have if it remained in the first excited state  $u_{11m}$  for the sphere in contraction. Here fifteen terms in eq. (32) leads to convergence.

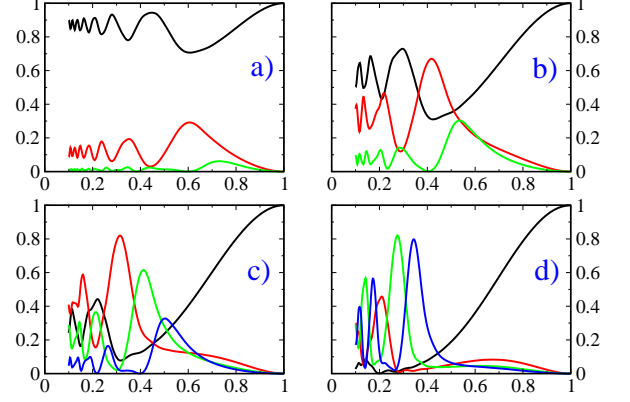


FIG. 1: (Color online) Transition probabilities versus  $\xi(t)$  for different values of velocity parameter  $\alpha$ : a)  $\alpha = -2$ ; b)  $\alpha = -4$ ; c)  $\alpha = -6$  and d)  $\alpha = -10$ . In each part the black curve shows  $|b_{11m}|^2$ , red one  $|b_{12m}|^2$ , green one  $|b_{13m}|^2$  and the blue one  $|b_{14m}|^2$ .

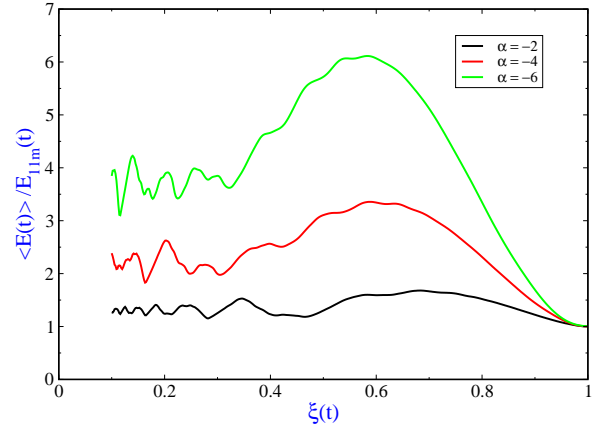


FIG. 2: (Color online) Ratio of the energy expectation value to the instantaneous first excited energy as a function of  $\xi(t)$  for three different values of velocity parameter.

We have plotted dimensionless radial probability density  $\rho_{ln}(\eta_{ln}, T_{ln}) = \lambda_{ln}^3 \eta_{ln}^2 |R(\eta_{ln}, T_{ln})|^2$  in fig. 3 for a particle initially in the state  $u_{0,5,0}$ , against dimensionless position coordinate  $\eta_{ln} = r/\lambda_{ln}$  at dimensionless time coordinate  $T_{ln}^{(0)} = \nu_{ln} t^{(0)}$  and in fig. 4 for a particle initially in the state (a)  $u_{0,15,0}$  and (b)  $u_{0,100,0}$ , against dimensionless time coordinate  $T_{ln} = \nu_{ln} t$  at dimensionless observation point  $\eta_{ln}^{(0)} = r^{(0)}/\lambda_{ln}$ ; where  $\lambda_{ln} = 2\pi a/x_{ln}$  and  $\nu_{ln} = E_{ln}/h$ . In our calculations  $r^{(0)} = 2a$  and  $t^{(0)} = (r^{(0)} - a)/u$ .

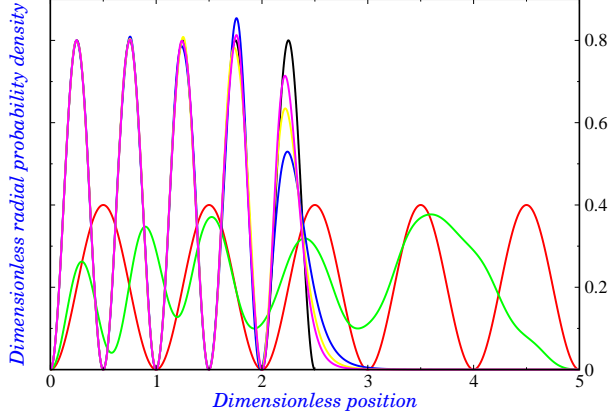


FIG. 3: (Color online) Dimensionless radial probability density  $\rho_{ln}(\eta_{ln}, T_{ln})$  for a particle initially in the state  $u_{0,5,0}$ , against dimensionless position coordinate  $\eta_{ln}$  at dimensionless time coordinate  $T_{ln}^{(0)}$ , for six different values of expansion rate;  $\alpha = 0$  (black curve),  $\alpha = 0.01\alpha_{ln}$  (red curve),  $\alpha = \alpha_{ln}$  (green curve),  $\alpha = 10\alpha_{ln}$  (blue curve),  $\alpha = 15\alpha_{ln}$  (yellow curve) and  $\alpha = 20\alpha_{ln}$  (magenta curve); where  $\alpha_{ln} = x_{ln}/2$ .

$T_1$  and  $T_2$  are dimensionless classical flight times from the front and back edges of the sphere to the dimensionless observation point  $\eta_{ln}^{(0)}$  for a particle in the state  $u_{lnm}$ .

#### IV. PROPAGATOR

One can construct the propagator as follows

$$\begin{aligned}
 |\Psi(t)\rangle &= S(t, t_0)|\Psi(t_0)\rangle \\
 &= \sum_{lnm} \sum_{l'n'm'} |\Psi_{lnm}(t)\rangle \langle \Psi_{lnm}(t) | S(t, t_0) | \Psi_{l'n'm'}(t_0)\rangle \\
 &\quad \times \langle \Psi_{l'n'm'}(t_0) | \Psi(t_0)\rangle \\
 &= \sum_{lnm} |\Psi_{lnm}(t)\rangle \langle \Psi_{lnm}(t) | \Psi(t_0)\rangle,
 \end{aligned}$$

where  $S(t, t_0)$  is the time evolution operator and we have used the fact that if the particle is in the state  $|\Psi_{lnm}\rangle$  at  $t_0$ , it remains in that state as the wall moves, *i.e.*,  $S(t, t_0)|\Psi_{lnm}(t_0)\rangle = |\Psi_{lnm}(t)\rangle$ . Now, we write this equation in the form

$$\Psi(\mathbf{r}, t) = \int_0^a dr' r'^2 \int d\Omega' K(\mathbf{r}, t; \mathbf{r}', t') \Psi(r', \theta', \phi', t') \quad (33)$$

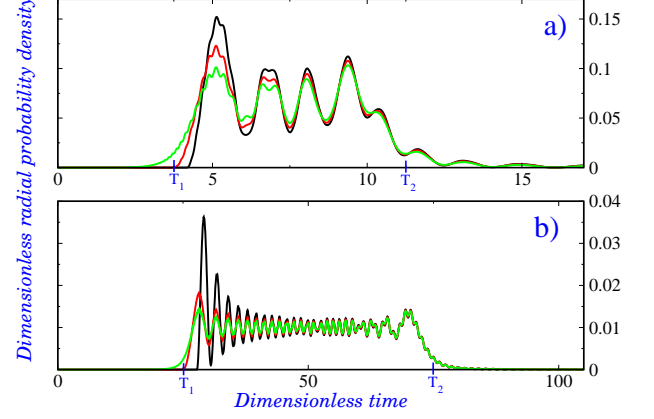


FIG. 4: (Color online) Dimensionless radial probability density  $\rho_{ln}(\eta_{ln}, T_{ln})$  for a particle initially in the state (a)  $u_{0,15,0}$  and (b)  $u_{0,100,0}$ , against dimensionless time coordinate  $T_{ln} = \nu_{ln}t$  at dimensionless observation point  $\eta_{ln}^{(0)}$ , for three different values of velocity parameter;  $\alpha = 0.9\alpha_{ln}$  (black curve),  $\alpha = \alpha_{ln}$  (red curve),  $\alpha = 2\alpha_{ln}$  (green curve); where  $\alpha_{ln} = x_{ln}/2$ .  $T_1$  and  $T_2$  are dimensionless classical flight times from the front and back edges of the sphere to the dimensionless observation point.

where we have introduced the propagator as,

$$\begin{aligned}
 K(\mathbf{r}, t; \mathbf{r}', t') &= \sum_{l=0}^{\infty} \sum_{n=1}^{\infty} \sum_{m=-l}^l \Psi_{lnm}(\mathbf{r}, t) \Psi_{lnm}^*(\mathbf{r}', t') \\
 &= \frac{2}{L^{3/2}(t) L^{3/2}(t')} \sum_{lnm} \frac{1}{[j_{l+1}(x_{ln})]^2} \\
 &\quad \times \exp \left[ \frac{i\mu u}{2\hbar} \left( \frac{r^2}{L(t)} - \frac{r'^2}{L(t')} \right) \right] \\
 &\quad \times \exp \left[ -\frac{i\hbar}{2\mu} \frac{x_{ln}^2}{a} \left( \frac{t}{L(t)} - \frac{t'}{L(t')} \right) \right] \\
 &\quad \times j_l \left( x_{ln} \frac{r}{L(t)} \right) j_l \left( x_{ln} \frac{r'}{L(t')} \right) \\
 &\quad \times Y_{lm}(\theta, \phi) Y_{lm}^*(\theta', \phi'). \quad (34)
 \end{aligned}$$

One sees when  $l = 0$ , eq. (7) reduces to eq. (4) of [2], *i.e.*,  $l = 0$  corresponds to a particle in a 1D box with the left wall at  $x = 0$  and the right wall in uniform motion. In order to have the relation

$$\Psi(x, t) = \int_0^a K_{1D}(x, t; x', 0) \Psi(x', 0) dx', \quad (35)$$

in 1D, we must write 1D propagator as

$$K_{1D}(x, t; x', t') = \frac{rr'}{4\pi} K(r, t; r', t') \\ \equiv \sum_{n=1}^{\infty} \frac{U_{0n}(r)}{\sqrt{4\pi}} \frac{U_{0n}(r')}{\sqrt{4\pi}}. \quad (36)$$

Preserving just the terms with  $l = 0$ , (34) leads

$$K(r, t; r', t') = \frac{2}{L^{3/2}(t)L^{3/2}(t')} \sum_{n=1}^{\infty} \frac{1}{[j_1(x_{0n})]^2} \\ \times \exp \left[ \frac{i\mu u}{2\hbar} \left( \frac{r^2}{L(t)} - \frac{r'^2}{L(t')} \right) \right] \\ \times \exp \left[ -\frac{i\hbar}{2\mu} \frac{x_{0n}^2}{a} \left( \frac{t}{L(t)} - \frac{t'}{L(t')} \right) \right] \\ \times j_0 \left( x_{0n} \frac{r}{L(t)} \right) j_0 \left( x_{0n} \frac{r'}{L(t')} \right) \\ \times \frac{1}{\sqrt{4\pi}} \frac{1}{\sqrt{4\pi}}, \quad (37)$$

where we have used  $Y_{00} = 1/\sqrt{4\pi}$ . The first two Bessel functions are

$$j_0(x) = \frac{\sin x}{x}, \\ j_1(x) = \frac{\sin x}{x^2} - \frac{\cos x}{x}, \quad (38)$$

thus  $x_{0n} = n\pi$  and  $j_1(x_{0n}) = (-1)^{n+1}/n\pi$ . Using these in eq. (37), we find

$$K(r, t; r', t') = \frac{1}{4\pi} \frac{1}{rr'} \frac{2}{\sqrt{L(t)L(t')}} \\ \times \exp \left[ \frac{i\mu u}{2\hbar} \left( \frac{r^2}{L(t)} - \frac{r'^2}{L(t')} \right) \right] \\ \times \sum_{n=1}^{\infty} \exp \left[ \frac{i\hbar}{2\mu} \frac{n^2\pi^2}{u} \left( \frac{1}{L(t)} - \frac{1}{L(t')} \right) \right] \\ \times \sin \left( n\pi \frac{r}{L(t)} \right) \sin \left( n\pi \frac{r'}{L(t')} \right). \quad (39)$$

Now from eq. (36) we obtain

$$K_{1D}(x, t; x', t') = \frac{2}{\sqrt{L(t)L(t')}} \\ \times \exp \left[ \frac{i\mu u}{2\hbar} \left( \frac{x^2}{L(t)} - \frac{x'^2}{L(t')} \right) \right] \\ \times \sum_{n=1}^{\infty} \exp \left[ \frac{i\hbar}{2\mu} \frac{n^2\pi^2}{u} \left( \frac{1}{L(t)} - \frac{1}{L(t')} \right) \right] \\ \times \sin \left( n\pi \frac{x}{L(t)} \right) \sin \left( n\pi \frac{x'}{L(t')} \right). \quad (40)$$

which is exactly eq. (32) of ref. [10] for the propagator of a particle in a 1D box. This equation can be written in a compact form as a combination of  $\vartheta_3$  functions [11].

## V. SUMMARY AND DISCUSSION

In this letter we found solutions of the Schrödinger equation for a particle confined in a hard spherical trap with a moving wall at constant velocity. We see in solutions (23), except for the phase factor  $\exp(-i \int dt E_{lnm}(t)/\hbar)$  which has no coordinate dependence, a coordinate-dependent phase  $\exp\left[\frac{i\mu}{2\hbar} u \frac{r^2}{L(t)}\right]$  appears. It has been shown that this factor leads to an effective quantum non-local interaction with the boundary: even though the particle is nowhere near the walls, it will be affected [1, 12].

From fig. 1, one sees that as the velocity of the wall increases, larger amounts of energy states other than the initial one, *i.e.*,  $u_{11m}$ , are mixed in. Fig. 2 shows that for rapid contraction, energy expectation value increases faster than the  $1/L^2(t)$  increase which would be obtained in a quasistatic contraction. These results are in agreement with the ones of ref. [3] obtained for a particle in an infinite square well with one wall in uniform motion. Confinement of the particle to a smaller region leads to enhancement of the energy expectation value. This can be explained by an application of the "old quantum theory" [13] or by uncertainty relations [14].

In the process of expansion, there are two characteristic times involved:  $t_e$ , over which the parameters of the system change appreciably, and  $t_i$ , representing the motion of the system itself. In our calculations,  $t_e = a/u$  and  $t_i = a/v_{ln}$ . Figure 3 shows that for  $t_e \gg t_i$  ( $u \ll v_{ln}$ ), the particle, initially in the state  $u_{0,5,0}$ , will end up in the corresponding state of the expanded well. This process characterizes an adiabatic one for which external conditions change gradually [15]. While, in the opposite limit, rapidly changing conditions prevent the system from adapting its configuration during the process, hence the probability density remains almost unchanged.

Noticing fig. 4, one sees a quasi-classical behavior in the high-energy limit [16] as the velocity of the wall increases. A non-monotonous increasing behavior of the density is seen for  $T < T_1$  only when  $u > v_{ln}$ , while for  $T > T_2$  a non-monotonous decreasing behavior is seen irrespective of the wall velocity. These results are in contrast to classical mechanics. The height of the first maximums decrease with  $u$ . The constructive interference with the reflected components from the wall for  $u < v_{ln}$  leads to this enhancement. Long time behavior of the density in the given observation point, is the same for all values of the wall velocity, which is not an unexpected result noticing the behavior of functions  $\Psi_{lnm}$  at long times.

Propagator of the problem was derived using the spectral decomposition.

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